# KPZ Universality conjectures and KPZ universality class

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1 Introduction : Random and ballistic deposition

So far, all the models studied linked to the KPZ univerality class are in (1 + 1) dimensions (one space & one time dimension)

- Blocks drop at rate one on every site of $\mathbb{Z}$
- First case : no interactions between the blocks, independant heights $(h_z)_{z \in \mathbb{Z}} : random deposition$
- Second case, the blocks stick to their neighbors, heights are no longer independant

The expected scales for the fluctuation of the depositions models are

<table>
<thead>
<tr>
<th>Random deposition</th>
<th>Ballistic deposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height expectation $\sim O(t)$</td>
<td>Height expectation $\sim O(t)$</td>
</tr>
<tr>
<td>Height fluctuations $\sim O(t^{1/2})$</td>
<td>Height fluctuations $\sim O(t^{1/3})$</td>
</tr>
<tr>
<td>No spatial correlation</td>
<td>Spatial correlations $\sim O(t^{2/3})$</td>
</tr>
</tbody>
</table>

The ballistic deposition model has three main characteristics :

**KPZ growth characteristics**

1. smoothing : the heights tend to homogeneize
2. slope dependant growth : when the slope is large, growth occurs more quickly
3. space time uncorrelated noise : independant blocks fall

2 Universality classes

2.1 Gaussian universality class :

CLT : By understanding normal varibales, the CLT gives intel on any average of random variable.

- By studying one object (BM/Gaussian variable), one can obtain results on a wide variety of models and quantities.
- The other way around, by showing things on discrete models, obtain results on the continuum limit
2.2 KPZ universality class:

Any growth model with these characteristics is expected to be in the KPZ universality class.

→ Main characteristics of the KPZ universality class is fluctuations of order $t^{1/3}$, and spatial correlations of order $t^{2/3}$.

Despite significant progress in the last decades, up to this point, the KPZ universality is not fully understood. (Cf. Universality conjectures, later in the talk)

3 The KPZ equation

General form of the KPZ equation

$$\partial_T h = \nu \partial_X^2 h + \lambda (\partial_X h)^2 + \sigma \dot{W}$$

- 3 parameters $\nu$, $\lambda$ and $\sigma$.
- By space and time rescaling, one can drop two of the parameters.
- KPZ equation → KPZ equations, the "KPZ equation" is in fact a one parameter family $\text{KPZ}(\gamma)$.

1) $\partial_X^2 h$: smoothing out 2) $(\partial_X h)^2$ Slope-dependent growth 3) $\dot{W}$ space-time white noise

It is expected that the KPZ universality involves both the scaling exponents as well as the long-time distributions, within geometry-dependent subclasses. These geometry-dependent subclasses depend on the initial profile. To illustrate that, ASEP.

4 The Asymmetric Simple Exclusion Process (ASEP)

4.1 The asymmetric exclusion process

Description of the model:

- On $\mathbb{Z}$, each site is either occupied ($\eta_x = 1$) or empty ($\eta_x = 0$)
- fix a (possibly random) initial configuration
- Each particles moves from $x$ to $x + 1$ at rate $1/2 < p < 1$
Each particle moves from $x$ to $x - 1$ at rate $q = 1 - p$.

Exclusion rule: any motion towards an occupied site is cancelled.

4.2 Various cases for the asymmetry

We denote by $\gamma = p - q$ the asymmetry of the system, therefore $p = (1 + \gamma)/2$, and $q = (1 - \gamma)/2$.

1. $\gamma = 0$, Symmetric Simple Exclusion Process (SSEP)
2. $\gamma = 1$, Totally Asymmetric Simple Exclusion Process (TASEP)
3. $0 < \gamma < 1$, Partially Asymmetric Simple Exclusion Process (PASEP)
4. $\gamma = \varepsilon^\beta\gamma$, with $\beta > 0$, Weakly Asymmetric Simple Exclusion Process (WASEP)

For now, we consider the PASEP, the case of the WASEP with $\beta = 1/2$ is linked to the KPZ equation and studied later on.

4.3 Height function: corner growth model

Given an ASEP configuration on $\mathbb{Z}$, one can build a height function $(h(x))_{x \in \mathbb{Z}}$

$$h(0) = 0 \quad \text{and} \quad h(x + 1) = \begin{cases} h(x) - 1 & \text{if } \eta_{x+1} = 1 \\ h(x) + 1 & \text{if } \eta_{x+1} = 0 \end{cases}.$$  

$\mapsto$ If a particle moves from $x$ to $x + 1$ in $\eta$, the local minimum of the function $h$ in $x$ becomes a local maximum.

$\mapsto$ If a particle moves from $x + 1$ to $x$ in $\eta$, vice-versa.

$$h_\gamma(t, x) = h_\gamma(0, x) + 2(N^-_x(t) - N^+_x(t)),$$

where $N^-_x(t)$ is the total number of particles that came to $x$ from $x - 1$ between the times 0 and $t$, and $N^+_x(t)$ is the total number of particles that came to $x$ from $x + 1$ between the times 0 and $t$.

5 Macroscopic limit and fluctuations

Question: what is the behavior of the system at a macroscopic scale?
**First solution** : given a smooth function \( H \) with bounded domain, study the behavior of
\[
\varepsilon \sum_{x \in \mathbb{Z}} H(\varepsilon x) \eta_x(C_\varepsilon t) \to \int_{\mathbb{R}} H(X) \rho(T, X) dX
\]
as \( \varepsilon \) goes to 0 ? \( \mapsto \) Weak formulation of local equilibrium.
We denote by \( X = x \varepsilon \) the macroscopic space variable and by \( T = C_\varepsilon t \) the macroscopic time.

**Second solution** : representation by the **height function**. The macroscopic profile of the corner growth model can be written as
\[
\tilde{h}(T, X) = \lim_{\varepsilon \to 0} \varepsilon h_{\gamma}(T, X, \frac{X}{\varepsilon}).
\]

**Hydrodynamic limit** : the macroscopic profile \( \tilde{h} \) is a weak solution to the inviscid Burgers equation
\[
\partial_T \tilde{h} = \frac{1 - (\partial_X \tilde{h})^2}{2}.
\]

**Particular solution with wedge initial condition** :
\[
\tilde{h}(T, X) = \frac{T \left(1 + (X/T)^2\right)}{2}.
\]

\( \mapsto \) Faire un dessin

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5.1 Back to the universality class : fluctuations

In accord with the KPZ fluctuations scale, one must consider the fluctuation around the hydrodynamic limit
\[
\tilde{f}_0(T, Z) = \lim_{\varepsilon \to 0} \varepsilon^{1/3} \left( h_{\gamma} \left( \frac{T}{\gamma \varepsilon}, \frac{Z}{\varepsilon^{2/3}} \right) - \frac{1}{\varepsilon} \tilde{h}(T, 0) \right).
\]
The compensating mean is indeed \( \tilde{h}(T, 0) \), because the scaling of the spatial fluctuation is less than the time rescaling. The fluctuation field around another macroscopic point \( X \) would be given by
\[
\tilde{f}_X(T, Z) = \lim_{\varepsilon \to 0} \varepsilon^{1/3} \left( h_{\gamma} \left( \frac{T}{\gamma \varepsilon}, \frac{X}{\varepsilon} + \frac{Z}{\varepsilon^{2/3}} \right) - \frac{1}{\varepsilon} \tilde{h}(T, X) \right),
\]
where \( Z \) is on a mesoscopic scale relatively to \( X \).

\( \mapsto \) Complété le dessin
5.2 Long time distributions and impact of the initial conditions

Getting back to the KPZ universality class, and long time distributions: Another strongly presumed universality feature of the universality class, additionally to the scaling exponents, would be the long-time distributions of the fluctuations. The field $f_0(T,0)$ is distributed in long time like the Tracy-Widom distribution.

5.3 Weak asymmetry and link to the KPZ equation

We now consider the case of the weak asymmetry with $\beta = \frac{1}{2}$. We now have

$$p = \frac{1}{2} + \varepsilon^{1/2}\gamma$$ and $$p = \frac{1}{2} - \varepsilon^{1/2}\gamma.$$

Then, the fluctuation field of the weakly asymmetric exclusion process should be solution to the KPZ($\gamma$) equation. (Bertini Giacomin '96)

More precisely this time, one considers the interface position equation

$$\overline{h}^w(T, X) = \lim_{\varepsilon \to 0} \varepsilon \cdot h_T\left( \frac{T}{\gamma \varepsilon^2}, \frac{X}{\varepsilon} \right).$$

Then, $\overline{h}^w(T, X)$ evolves according to the Burgers equation

$$\partial_T \overline{h}^w = \frac{1}{2} \Delta \overline{h}^w + \frac{1 - (\partial_X \overline{h}^w)^2}{2}.$$

In (Bertini Giacomin '96), it is proved that for the weakly asymmetric corner growth model, the fluctuations evolve according to the KPZ equation, i.e. that letting

$$\overline{f}_0(T, Z) = \lim_{\varepsilon \to 0} \varepsilon^{1/2} \left( h_T\left( \frac{T}{\gamma \varepsilon^2}, \frac{Z}{\varepsilon} \right) - \frac{1}{\varepsilon} \overline{h}^w(T, 0) \right),$$

the function $\overline{f}_0(T, Z)$ is solution to the KPZ equation with parameter $\gamma$.

6 Weak and Strong universality conjectures, Rescaling operator, Link with the Wilkinson Edwards universality class

Faire un dessin